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The distributions of the determinant of fixed-trace ensembles of real-symmetric and of Hermitian random matrices

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Abstract

Probability densities of the determinant are obtained for fixed-trace ensembles of $N \times N$ Hermitian and real-symmetric matrices from the exact determinant densities of the Gaussian orthogonal ensemble (GOE) and of the Gaussian unitary ensemble (GUE), respectively.

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1. Introduction

Random matrix theory (RMT), which was introduced in multivariate statistical analysis by Wishart in 1928 [1], is still of primary importance in that field [1–4]. RMT finds considerable use in various branches of physics, for instance in nuclear physics, quantum chaology and in several areas of quantum field theory [5–10]. In a short summary of the main historical developments of random matrix theory, Forrester *et al* [5] recall that the universality of correlations of eigenvalues of large random matrices on the scale of the average level spacing is at the root of the physical importance of RMT [5–10], a theory which has further attracted a great deal of interest because of the mathematical challenges it poses.

The distributions of random determinants were first investigated in connection with multidimensional statistical analysis [11–15]. Besides the importance of determinants as mathematical tools in RMT, questions about random determinants and their applications are quite naturally raised in that domain too. Motivated by conjectures in number theory for the zeros of the Riemann ζ-function, recent investigations of random determinants in RMT concentrated on the characteristic polynomial, $\det(x\mathbf{I}_N - \mathbf{H}_N)$, on the asymptotic universality of expressions of moments of all kinds of its distribution and on the universality of correlation functions containing an arbitrary number of products and ratios of distinct characteristic polynomials with respect to various ensembles (see for instance [16–24]).

The problem of the distribution of the determinant D of a $N \times N$ random matrix $\mathbf{H}_N$ from random matrix ensembles (RMEs) considered in RMT, although narrower in scope, is nevertheless of definite mathematical and physical interest as mentioned hereafter for ensembles of real symmetric and of Hermitian random matrices on which we shall concentrate. This determinant problem was considered only recently for the three classical Gaussian ensembles [25–27]. The probability densities of matrices from these Gaussian ensembles are recalled to be proportional to $\exp[-\mathrm{tr}(V(\mathbf{H}_N))]$, with an harmonic potential $V(\mathbf{H}_N) \propto \mathbf{H}_N^2$, where 'tr' means trace. Matrices are for instance real symmetric for the Gaussian orthogonal ensemble (GOE) and Hermitian for the Gaussian unitary ensemble (GUE) [3]. The Mellin transforms of the determinant distributions were derived for $N \times N$ matrices from the GUE [25] and from the GOE [26], and the moments $\langle D^m \rangle_N$ were deduced whatever m and N [25, 26]. Very recently, alternative and attractive expressions for these moments were calculated among other things by Andrews *et al* [27]. Explicit expressions were finally found for the GUE and the GOE determinant distributions [25, 26] by an inversion of their Mellin transforms except for the case of $2k \times 2k$ matrices from the GOE. Cicuta and Mehta [28] calculated the probability densities of the determinant of Gaussian $N \times N$ random matrices, without symmetry specifications, whose elements are either real or complex or quaternions; the real random variables entering these matrix elements being independent and identically distributed according to a Gauss law with mean zero. The average values of the determinant of random real matrices $\mathbf{H}_N$ whose entries are identically distributed, correlated Gaussian random variables with mean zero were reported for various covariance structures [29].

The properties of eigenvalues of RMEs can be interpreted in 2D from the equilibrium characteristics of a gas of N identical point charges on a line ([5–10] and references therein), often referred to as a log-gas [30], which interact via a logarithmic Coulomb potential and are confined by an external potential without singularities. Besides their interest per se, the Mellin transforms of the determinant distributions [25–27] are further the partition functions of log-gases whose harmonic confinement potential includes a supplementary logarithmic contribution. Such gases were considered for instance in the context of quantum chromodynamics and of mesoscopic electron transport [31, 32]. Further, the general linear statistic problem ([33–35] and references therein) deals with physical quantities Q which are given by sums $Q = \sum_{k=1}^{N} f(\lambda_k)$ over the eigenvalues λ_k of a random matrix, where $f(x)$ may depend nonlinearly on x. Within the log-gas interpretation, the statistic of the potential at the origin is obtained for $f(x) = -\log(|x|)$, that is for $Q = -\log(|D|)$. It may thus be investigated from the Mellin transform of the distribution of D and was shown to be asymptotically Gaussian for the GOE and the GUE [26] as expected from general theorems on linear statistics [35].

The focus of the present paper shall be on the distributions of the determinant of real symmetric and of Hermitian $N \times N$ random matrices from fixed-trace ensembles (FTEs). Fixed-trace ensembles of random matrices were first defined by Rosenzweig and Bronk ([36, 37] as quoted in [7], chapter 19), by $\mathrm{tr}\left(\mathbf{H}_N^{+}\mathbf{H}_N\right) = \text{constant}$ with no other constraint, where $\mathbf{H}_N^{+}$ is the Hermitian conjugate of $\mathbf{H}_N$. Akemann *et al* [38] describe specific physical features of FTEs which are due to the interaction among eigenvalues introduced through a constraint. Advantage was taken previously of the connection between Gaussian RMEs and the FTEs ([38–40] and section 2) to derive the exact densities of states of FTEs [38, 39]. In the same vein, we calculate here the determinant distributions of FTEs from those of the corresponding Gaussian ensembles.

2. The connection between Gaussian and fixed-trace ensembles of random matrices

2.1. Generalities

The number of distinct real random variables which are necessary to construct a $N \times N$ Gaussian matrix $\mathbf{H}_N$ from the GOE ($\beta = 1$) or from the GUE ($\beta = 2$) is $N_p = N + \beta \frac{N(N-1)}{2}$, the N_p variables $H_N(i, j)$ being independent Gaussian distributed $N\left(0, \frac{\sigma^2(1+\delta_{ij})}{2}\right)$ random variables ($i = 1, \ldots, N$, $j = i, \ldots, N$). The notation $P_{N,\beta}(Y_X)$ and $P^X_{N,\beta}(Y)$, with $X = F, G$, is used throughout the text for the probability density of the random variable Y for fixed-trace ensembles (FTE(β)) and Gaussian ensembles, respectively.

A one-to-one correspondence is first established between a random vector $\mathbf{U}_{N_p}$ uniformly distributed on the surface of the unit N_p-dimensional sphere and a matrix $\mathbf{S}_N(\beta)$ in such a way that the usual norm of $\mathbf{U}_{N_p}$ is equal to the norm of $\mathbf{S}_N(\beta)$, $\|\mathbf{S}_N(\beta)\| = \left(\mathrm{tr}\left(\mathbf{S}^+_N(\beta)\mathbf{S}_N(\beta)\right)\right)^{1/2} = 1$ [39, 40]. The random matrices $\mathbf{S}_N(\beta)$ of the ensembles FTE(β) have a norm which is taken here as 1 without loss of generality. For $N = 2$, uniform distributions on the surface of unit three- and four-dimensional spheres yield for instance the following matrices for $\beta = 1$ and 2, respectively:

$$\mathbf{S}_2(1) = \begin{bmatrix} U(1) & \frac{U(2)}{\sqrt{2}} \\ \frac{U(2)}{\sqrt{2}} & U(3) \end{bmatrix} \qquad \mathbf{S}_2(2) = \begin{bmatrix} U(1) & \frac{(U(2)+\mathrm{i}U(3))}{\sqrt{2}} \\ \frac{(U(2)-\mathrm{i}U(3))}{\sqrt{2}} & U(4) \end{bmatrix} \tag{1}$$

where $U(k)$ is the kth component of the unit vector $\mathbf{U}_{N_p}$.

Spherical random vectors are characterized by their stochastic representation [40, 41]

$$\mathbf{X}_{N_p} \stackrel{d}{=} R\mathbf{U}_{N_p}$$

where R is some non-negative random variable independent of $\mathbf{U}_{N_p}$. Associating matrices of the same symmetry with both vectors $\mathbf{X}_{N_p}$ and $\mathbf{U}_{N_p}$, we obtain

$$\mathbf{M}_N(\beta) \stackrel{d}{=} R\mathbf{S}_N(\beta) \tag{2}$$

where the matrix $\mathbf{M}_N(\beta)$ is constructed from $\mathbf{X}_{N_p}$ in the same way as $\mathbf{S}_N(\beta)$ is from $\mathbf{U}_{N_p}$. For the GOE and the GUE, whose matrices $\mathbf{G}_N(\beta)$ are distributed according to

$$P_{N,\beta}(\mathbf{G}_N) = \alpha_{N,\beta} \exp\left[-\frac{\mathrm{tr}\left(\mathbf{G}^2_N\right)}{2\sigma^2}\right] \tag{3}$$

where $\alpha_{N,\beta}$ is a normalization constant (see for instance [7]), R^2/σ^2 is χ^2 distributed with N_p degrees of freedom as $R^2 = \mathrm{tr}\left(\mathbf{G}^2_N\right)$ is the sum of the squares of N_p independent $N(0, \sigma^2)$ Gaussian variables. Equation (2) with $\mathbf{M}_N(\beta) = \mathbf{G}_N(\beta)$ yields a direct method of calculation of a given property of $\mathbf{S}_N(\beta)$, and thus of that of any spherical matrix, once it is known for the Gaussian ensemble of the same symmetry.

Monte Carlo simulations were performed to generate matrices $\mathbf{S}_N(\beta)$ for $\beta = 1, 2$ and to compare their determinant distributions to those calculated below. The random vector $\mathbf{U}_{N_p}$ was obtained from the classical stochastic representation, $\mathbf{U}_{N_p} \stackrel{d}{=} \frac{\mathbf{G}_{N_p}}{\|\mathbf{G}_{N_p}\|}$ [42], where $\mathbf{G}_{N_p}$ is a N_p-dimensional vector whose components are iid $N(0, 1)$ Gaussian variables. The $\mathbf{G}_{N_p}$ components were generated by the standard Box–Müller method [42].

After having summarized recent literature results on the determinant distributions of Gaussian ensembles [25, 26], we establish general relations between the determinant probability densities of Gaussian RMEs of a given symmetry and those of the FTEs of the same symmetry from which we derive the FTEs determinant densities.

2.2. Determinant distributions of the GOE and of the GUE

The joint distributions of eigenvalues for the previous Gaussian ensembles are [7]

$$P_{N,\beta}(\lambda_1,\ldots,\lambda_N) = C_{N,\beta}\exp\left(-\frac{1}{2\sigma^2}\left[\sum_{k=1}^{N}\lambda_k^2\right]\right)\left[\prod_{1\leqslant j<k\leqslant N}|\lambda_j-\lambda_k|^{\beta}\right]. \tag{4}$$

The Mellin transforms of the even and odd parts of the distributions, $P_{N,\beta}(D_G)$, of the determinant D_G of the GO(U)E

$$P^{\pm}_{N,\beta}(D_G = \tfrac{1}{2}[P_{N,\beta}(D_G) \pm P_{N,\beta}(-D_G)] \tag{5}$$

are then calculated separately [25]. They are obtained from

$$M^{G,\pm}_{N,\beta}(s) = \int_0^{\infty} D^{s-1}P^{\pm}_{N,\beta}(D)\,\mathrm{d}D = \frac{1}{2}\int_{\mathbb{R}^N} P_{N,\beta}(\lambda_1,\ldots,\lambda_N)\prod_{k=1}^{N}|\lambda_k|^{s-1}\varepsilon^{\pm}(\lambda_k)\,\mathrm{d}\lambda_k \tag{6}$$

with $\varepsilon^+(x) = 1, \varepsilon^-(x) = \mathrm{sgn}(x)$.

The Mellin transform of Meijer's G-function $G^{m,n}_{p,q}\left(\alpha x^2\big|^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right)(0 \leqslant m \leqslant q, 0 \leqslant n \leqslant p)$, in which any product $\prod_{j=i}^{k}$is taken as 1 whenever $k < i$, is [43, 44]

$$\int_0^{\infty} x^{s-1}G^{m,n}_{p,q}\left(\alpha x^2\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right)\mathrm{d}x = \frac{\alpha^{\frac{-s}{2}}}{2}\left[\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{s}{2}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-\frac{s}{2}\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{s}{2}\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_j-\frac{s}{2}\right)}\right]. \tag{7}$$

For N odd, the determinant distribution$P_{N,\beta}(D_G)$ is symmetric and is proportional to a single Meijer's G-function defined solely by the N parameters $b_j^{(\beta)}(j = 1,\ldots,N)$ [25, 26] :

$$P_{2p+1,1}(D_G) = K_{2p+1,1}G^{2p+1,0}_{0,2p+1}\left(\frac{D_G^2}{(2\sigma^2)^N}\left|0, b_2^{(\beta)}, b_3^{(\beta)},\ldots,b_{2p+1}^{(\beta)}\right.\right) \tag{8}$$

with

$$b_1^{(\beta)} = 0$$
$$b_j^{(1)} = \frac{1}{2}\left\lfloor\frac{j-1}{2}\right\rfloor + \frac{1}{4} \qquad (2 \leqslant j \leqslant N) \qquad b_j^{(2)} = \left\lfloor\frac{j}{2}\right\rfloor \qquad (j = 1,\ldots,N) \tag{9}$$

where $\lfloor x\rfloor$ denotes the largest integer $\leqslant x$, and the normalization constant $K_{N,\beta}$ is given by

$$[K_{N,\beta}]^{-1} = (2\sigma^2)^{N/2}\prod_{j=1}^{N}\Gamma\left(\frac{1}{2}+b_j^{(\beta)}\right). \tag{10}$$

For the GUE with N even, the asymmetric determinant distribution is a linear combination of two Meijer's G-functions [25]:

$$P_{2p,2}(D_G) = K_{2p,2}\left[G^{2p,0}_{0,2p}\left(\frac{D_G^2}{(2\sigma^2)^N}\left|0,1,1,2,2,\ldots,p-1,p-1,p\right.\right)\right.$$
$$\left. + (-1)^p\,\mathrm{sign}(D_G)G^{2p,0}_{0,2p}\left(\frac{D_G^2}{(2\sigma^2)^N}\left|\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\ldots,p-\frac{1}{2},p-\frac{1}{2}\right.\right)\right]. \tag{11}$$

The Mellin transform of the odd part of the latter distribution is seen to be defined by parameters

$$b_j^{(2-)} = \frac{1}{2} + \left\lfloor\frac{j-1}{2}\right\rfloor \qquad (j = 1,\ldots,N). \tag{12}$$

2.3. Relation between determinant distributions of Gaussian RMEs and of FTEs

As the maximum of $\prod_{k=1}^{N}|\lambda_k|$ subject to $\sum_{k=1}^{N}\lambda_k^2 = 1$ is $N^{-N/2}$ we rescale the determinant:

$$D = N^{N/2}\left[\prod_{k=1}^{N}\lambda_k\right] \tag{13}$$

so that

$$P^F_{N,\beta}(D) = 0 \qquad \text{for} \quad |D| \geqslant 1 \tag{14}$$

whatever N, where $P^F_{N,\beta}(D)$ is the determinant density of FTE(β). The determinant distributions $P^G_{N,\beta}(D)$ and $P^F_{N,\beta}(D)$ are related through

$$P^G_{N,\beta}(D) = \frac{2}{2^{\frac{N_p}{2}-1}\sigma^{N_p}\Gamma\left(\frac{N_p}{2}\right)}\int_{|D|^{\frac{1}{N}}}^{\infty} P^F_{N,\beta}\left(\frac{D}{r^N}\right) r^{N_p-N-1}\exp\left(-\frac{r^2}{2\sigma^2}\right)\mathrm{d}r. \tag{15}$$

Equation (15) expresses simply the total density as a sum of densities $P^F_{r,N,\beta}(D)$ associated with the surfaces of spheres of radii r, namely $P^F_{r,N,\beta}(D) = P^F_{N,\beta}(D/r^N)/r^N$, with varying weights $f(r)\,\mathrm{d}r$. From condition (14), only the spheres with $r \geqslant |D|^{1/N}$ contribute to the total density $P^G_{N,\beta}(D)$ for a given D. Finally, the weight $f(r)\mathrm{d}r$ of a sphere of radius r is deduced from the distribution of $r^2 = \mathrm{tr}\left(G_N^2\right)/\sigma^2$ which is a χ^2 distribution with N_p degrees of freedom for the Gaussian ensembles.

3. Probability density of the determinant for FTE(β)

The distribution $P^F_{2,\beta}(D)$ is easily derived from the eigenvalues $\lambda_\pm = \frac{1}{2}\left[x \pm \sqrt{(2-x^2)}\right]$ of the random matrices $\mathbf{S}_2(\beta)$ (equation (1)), where $x = U(1) + U(2+\beta)$. The distribution of $x \overset{d}{=} U(1)\sqrt{2}$ is deduced from that of the first component of the vector $\mathbf{U}_{\beta+2}$ uniformly distributed on the surface of the unit($\beta+2$)-dimensional sphere (appendix A of [40]). The distribution of x ($-\sqrt{2} \leqslant x \leqslant \sqrt{2}$) is uniform for $\beta = 1$, $P_{2,1}(x) = 1/2\sqrt{2}$ while $P_{2,2}(x) = \sqrt{2-x^2}/\pi$ for $\beta = 2$. As $D = 2\lambda_+\lambda_- = x^2 - 1$, we obtain ($|D| < 1$)

$$P^F_{2,\beta}(D) = C_{2,\beta}\frac{(1-D)^{\beta/2}}{\sqrt{(1-D^2)}} \tag{16}$$

for $\beta = 1, 2$ ($C_{2,1} = 1/2\sqrt{2}$, $C_{2,2} = 1/\pi$). The distribution $P^F_{2,1}(D)$ allows us, for instance, to calculate the distribution of the determinant of a 2×2 GOE matrix in a very simple way [26]. To calculate the determinant distribution for larger values of N, we take first the Mellin transform of the odd and even parts of the distributions (equation (5)) obtained from both members of equation (15) (with $\sigma = 1/\sqrt{2}$):

$$M^{F\pm}_{N,\beta}(s) = M^{G\pm}_{N,\beta}(s)\frac{\Gamma\left(\frac{N_p}{2}\right)}{\Gamma\left(\frac{N(s-1)+N_p}{2}\right)}. \tag{17}$$

Some integrals related to $M^F_{N,\beta}(s)$ are discussed in appendix A. As the determinant defined by equation (13) is $N^{N/2}$ times the usual determinant, the Mellin transforms $M^G_{N,\beta}(s)$ of distributions equations (8) and (11) must first be multiplied by $N^{N(s-1)/2}$ before being inserted in equation (17). The multiplication formula is then applied to decouple N and s in the gamma function of the denominator of equation (17), namely

$$\Gamma\left(N\left(\frac{s}{2}+\beta\frac{(N-1)}{4}\right)\right) = (2\pi)^{\frac{(1-N)}{2}} N^{N\left(\frac{s}{2}+\beta\frac{(N-1)}{4}\right)-\frac{1}{2}}\prod_{j=1}^{N}\Gamma\left(\frac{s}{2}+a_j^{(\beta)}\right) \tag{18}$$

where $a_j^{(\beta)}$ $(j = 1, \ldots, N)$ is defined as

$$a_j^{(\beta)} = \beta\frac{(N-1)}{4} + \frac{j-1}{N} \qquad (j = 1, \ldots, N). \tag{19}$$

When the Mellin transforms $M_{N,\beta}^{G\pm}(s)$ of the odd and even parts, $P_{N,\beta}^{\pm}(D_G)$, of the determinant distributions $P_{N,\beta}(D_G)$ of Gaussian RMEs are known products of gamma functions as described in section 2.2, we get

$$M_{N,\beta}^{F+}(s) = \frac{C_{N,\beta}^{F+}}{2}\left[\frac{\prod_{j=1}^{N}\Gamma\left(\frac{s}{2}+b_j^{(\beta)}\right)}{\prod_{j=1}^{N}\Gamma\left(\frac{s}{2}+a_j^{(\beta)}\right)}\right] \tag{20}$$

where

$$C_{N,\beta}^{F+} = \left[\frac{\prod_{j=1}^{N}\Gamma\left(\frac{1}{2}+a_j^{(\beta)}\right)}{\prod_{j=1}^{N}\Gamma\left(\frac{1}{2}+b_j^{(\beta)}\right)}\right] \tag{21}$$

is obtained from $M_{N,\beta}^{F+}(1) = \frac{1}{2}$. The Mellin transform of the odd part is

$$M_{2p,2}^{F-}(s) = \frac{C_{2p,2}^{F-}}{2}\left[\frac{\prod_{j=1}^{2p}\Gamma\left(\frac{s}{2}+b_j^{(2-)}\right)}{\prod_{j=1}^{2p}\Gamma\left(\frac{s}{2}+a_j^{(2)}\right)}\right] \tag{22}$$

for $\beta = 2$ and N even (see equation (12)). In summary

(a) for $N = 2p+1 \quad (-1 < D < 1)$:

$$P_{N,\beta}^{F}(D) = C_{2p+1,\beta}^{F+}G_{2p+1,2p+1}^{2p+1,0}\left(D^2\left|\begin{matrix}a_j^{(\beta)}, j = 1, \ldots, 2p+1\\ b_j^{(\beta)}, j = 1, \ldots, 2p+1\end{matrix}\right.\right) \tag{23}$$

(b) for $N = 2p \quad (p \geqslant 2, -1 < D < 1)$:

$$P_{N,2}^{F}(D) = C_{2p,2}^{F+}\left[G_{2p,2p}^{2p,0}\left(D^2\left|\begin{matrix}a_j^{(2)}, j = 1, \ldots, 2p\\ b_j^{(2)}, j = 1, \ldots, 2p\end{matrix}\right.\right)\right.$$
$$\left. + (-1)^p \operatorname{sign}(D) G_{2p,2p}^{2p,0}\left(D^2\left|\begin{matrix}a_j^{(2)}, j = 1, \ldots, 2p\\ b_j^{(2-)}, j = 1, \ldots, 2p\end{matrix}\right.\right)\right]. \tag{24}$$

From the definition of Meijer's G-function (equation (7)), we calculate

$$\int_0^1 D^s G_{N,N}^{N,0}\left(D^2\left|\begin{matrix}a_1^{(\beta)}, \ldots, a_N^{(\beta)}\\ b_1^{(\beta)}, \ldots, b_N^{(\beta)}\end{matrix}\right.\right)\mathrm{d}D = \frac{1}{2}\prod_{j=1}^{N}\frac{\Gamma\left(\frac{s+1}{2}+b_j^{(\beta)}\right)}{\Gamma\left(\frac{s+1}{2}+a_j^{(\beta)}\right)}. \tag{25}$$

Equation (25) also holds when $b_j^{(\beta)}$ is replaced by $b_j^{(\beta-)}$. The even moments $\left\langle D_F^{2k}\right\rangle_{N,\beta}$ of the determinant distribution $P_{N,\beta}(D_F)$ (with N odd for $\beta = 1$) are then given by

$$\left\langle D_F^{2k}\right\rangle_{N,\beta} = \int_{-1}^{1} D_F^{2k} P_{N,\beta}(D_F)\,\mathrm{d}D_F = \prod_{j=1}^{N}\left[\frac{\left(\frac{1}{2}+b_j^{(\beta)}\right)_k}{\left(\frac{1}{2}+a_j^{(\beta)}\right)_k}\right] \tag{26}$$

where $(a)_k$ is the Pochhammer symbol, $(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$. The product in equation (26) simplifies in general as some common factors arise in the numerator and in the denominator. Moments obtained from Monte-Carlo simulations for $\beta = 1, 2$, $p = 1, \ldots, 4$, $k = 1, \ldots, 5$ are found to agree with equation (26) (see table 1 for $\beta = 1$).

Numerical calculations of Meijer's G-functions can be performed from residue calculations [25, 43–45]. Monte-Carlo simulations were used here to calculate the

Table 1. Comparison of the moments $\langle D_F^{2k}\rangle_{N,1}$ obtained by Monte Carlo simulations, $2k$ (s), from n $N \times N$ matrices with the moments, $2k$ (c), calculated from equation (26) with $a_j^{(1)}$ and $b_j^{(1)}$ $(j = 1, \ldots, N)$ given by equations (19) and (9), respectively.

$2k$	$N = 3\,(n = 2 \times 10^7)$	$N = 5\,(n = 2 \times 10^7)$	$N = 7\,(n = 2 \times 10^7)$	$N = 9\,(n = 10^7)$
2 (s)	0.21089	4.3816×10^{-2}	8.512×10^{-3}	1.569×10^{-3}
2 (c)	0.2109375	4.3817584×10^{-2}	8.5104375×10^{-3}	1.5678546×10^{-3}
4 (s)	0.10009	7.9962×10^{-3}	5.117×10^{-4}	2.773×10^{-5}
4 (c)	0.10011291	7.9964654×10^{-3}	5.1147344×10^{-4}	2.7655273×10^{-5}
6 (s)	6.0992×10^{-2}	2.4456×10^{-3}	6.741×10^{-5}	1.404×10^{-6}
6 (c)	6.1006308×10^{-2}	2.4455811×10^{-3}	$6.74246102 \times 10^{-5}$	1.3928135×10^{-6}
8 (s)	4.206×10^{-2}	9.834×10^{-4}	1.371×10^{-5}	1.314×10^{-7}
8 (c)	4.2070155×10^{-2}	9.8310601×10^{-4}	1.3761447×10^{-5}	1.2873710×10^{-7}
10 (s)	3.124×10^{-2}	4.689×10^{-4}	3.678×10^{-6}	1.837×10^{-8}
10 (c)	3.1242310×10^{-2}	4.6852523×10^{-4}	3.7180134×10^{-6}	1.7739068×10^{-8}

$G_{N,N}^{N,0}\left(x\middle|{}^{a_j, j=1,\ldots,N}_{b_j, j=1,\ldots,N}\right)$ needed for the distributions given by equations (23) and (24) for $N > 2$ as $a_j > b_j$. Indeed, the probability density of the product $Y = \prod_{j=1}^{N} X_j$ of N independent beta random variables of the first kind X_j whose probability densities are

$$f_j(x_j) = \frac{\Gamma(a_j+1)}{\Gamma(b_j+1)\Gamma(a_j-b_j)} x_j^{b_j}(1-x_j)^{a_j-b_j-1}$$
$$(0 < x_j < 1, \quad a_j > b_j > 0, \quad j = 1, \ldots, N) \tag{27}$$

is proportional to $G_{N,N}^{N,0}\left(y\middle|{}^{a_j, j=1,\ldots,N}_{b_j, j=1,\ldots,N}\right)$ (p 104 of [43], section 2.3.4 of [44]).

3.1. Real-symmetric case: FTE(1) with $N = 2p + 1$

The symmetric distribution $P_{2p+1,1}^F(D)$ is given by equation (23) with $a_j^{(1)} = \frac{(N-1)}{4} + \frac{j-1}{N}$ (equation (19)), $b_1^{(1)} = 0$ and $b_j^{(1)} = \frac{1}{2}\left\lfloor \frac{j-1}{2} \right\rfloor + \frac{1}{4}$ $(2 \leqslant j \leqslant N)$ (equation (9)). The moments $\langle D_F^m \rangle_{N,1}$ are related to the moments $\langle D_G^m(\sigma)\rangle_{N,1}$ calculated in the Gaussian case (equations (C.1), (C.2), (C.3) of appendix C of [26] with $\sigma = 1/\sqrt{2}$) by

$$\langle D_F^m \rangle_{2p+1,1} = N^{\frac{Nm}{2}} \frac{\Gamma\left(\frac{N(N+1)}{4}\right)}{\Gamma\left(\frac{N(N+1)}{4} + \frac{Nm}{2}\right)} \langle D_G^m(1/\sqrt{2})\rangle_{N,1}. \tag{28}$$

The variance of the distribution $P_{2p+1,1}^F(D)$ is thus

$$\langle D_F^2 \rangle_{2p+1,1} = \frac{(N+2)}{3}\left(\frac{N}{8}\right)^N \frac{\left[\prod_{j=1}^{p+1}(p+1+j)^2\right]}{\left(\frac{N(N+1)}{4}\right)_N}. \tag{29}$$

The distribution $P_{3,1}^F(D)$ is deduced from equation (23)

$$P_{3,1}^F(D) = \frac{16}{9\sqrt{6\pi}} G_{3,3}^{3,0}\left(D^2 \middle| {}^{\frac{1}{2}, \frac{5}{6}, \frac{7}{6}}_{0, \frac{1}{4}, \frac{3}{4}}\right) = \frac{16}{9\pi\sqrt{6}}\left((3\sqrt{2})\,{}_3F_2\left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; \frac{1}{4}, \frac{3}{4}; D^2\right)\right.$$
$$- 2(3)^{1/4}|D|^{1/2}\,{}_3F_2\left(\frac{1}{12}, \frac{5}{12}, \frac{3}{4}; \frac{1}{2}, \frac{5}{4}; D^2\right)$$
$$\left. - 3^{-5/4}|D|^{3/2}\,{}_3F_2\left(\frac{7}{12}, \frac{11}{12}, \frac{5}{4}; \frac{3}{2}, \frac{7}{4}; D^2\right)\right) \tag{30}$$

in excellent agreement with the simulation results (figure 1) as also seen for $N = 3$ to 11.

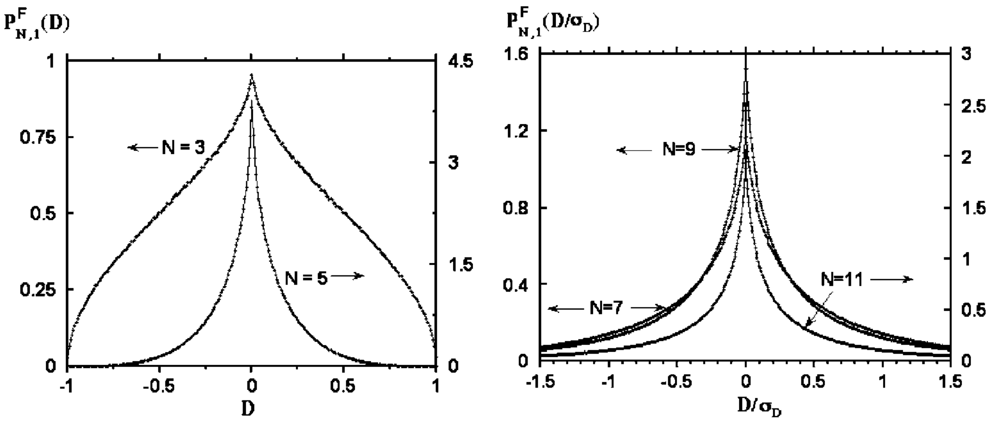

Figure 1. Determinant distributions $P^F_{N,1}(D)$ obtained for odd N from Monte Carlo simulations with 2×10^7 matrices (crosses). The theoretical distributions (solid lines) are calculated from equation (30) for $N = 3$ and from equations (23) and (27) for $N = 5$ to 11, respectively. The distribution of $\frac{D}{\sigma_D}$ ($\sigma_D = \langle D^2\rangle^{\frac{1}{2}}$) is shown for clarity for $N = 7, 9, 11$.

3.2. Hermitian case: FTE(2)

For $N = 3$, Meijer's G-function simplifies to $G^{2,0}_{2,2}\left(D^2\middle|^{\frac{4}{3}\ \frac{5}{3}}_{0\ 1}\right)$ which is calculated for instance from theorem 2.7 of Mathai [44]:

$$P^F_{3,2}(D) = \frac{35}{54}(1 - D^2)F\left(\frac{1}{3}, \frac{2}{3}; 2; 1 - D^2\right) \tag{31}$$

where $F(a, b, c; z)$ is an hypergeometric function and even moments are $\langle D^{2k}\rangle_{3,2} = \frac{35}{54}\frac{\Gamma(k+1/2)\Gamma(k+3/2)}{\Gamma(k+11/6)\Gamma(k+13/6)}$. Figure 2 shows $P^F_{N,2}(D)$ for $3 \leqslant N \leqslant 6$.

For $N = 2p+1$, the even moments (equation (26)) can be rewritten in the Hermitian case as

$$\langle D^{2k}\rangle_{2p+1,2} = \frac{N^{Nk}\Gamma\left(\frac{N^2}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(kN + \frac{N^2}{2}\right)\sqrt{\pi}}\prod_{j=1}^{k}\left[\frac{\Gamma\left(p + j + \frac{1}{2}\right)}{\Gamma\left(j + \frac{1}{2}\right)}\right]^2. \tag{32}$$

When $N = 2p$, the odd moments are

$$\langle D^{2k+1}\rangle_{2p,2} = \frac{(-1)^p}{\left(\frac{1}{2}\right)_p}\prod_{j=1}^{2p}\left[\left(b_j^{(2-)}\right)_{k+1}\frac{\Gamma\left(\frac{1}{2} + a_j^{(2)}\right)}{\Gamma\left(k + 1 + a_j^{(2)}\right)}\right]. \tag{33}$$

When $N = 2p + 1 \to \infty$, the variance $\langle D^2\rangle_{N,2}$ varies as $N\exp(-N)$ and the moments $r_{2k,2}$

$$r_{2k,2} = \frac{\langle D^{2k}\rangle_{N,2}}{\langle D^2\rangle^k_{N,2}} \approx \left(\frac{p}{e}\right)^{k(k-1)}\left(\frac{\pi}{8}\right)^k\frac{(2k)!}{(k!)\left[\prod_{j=1}^k \Gamma^2(j + 1/2)\right]} \qquad (k \geqslant 1) \tag{34}$$

increase rapidly for $k \geqslant 2$. The moment variation evidences the heavy-tailed nature of the asymptotic distribution and explains the rapid change of shape of $P^F_{N,2}(D)$ with N as seen in figure 2. A similar change is observed for the GOE (figure 1). For a lognormal distribution,

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}}\exp\left(-\frac{(\ln(x/m))^2}{2\sigma^2}\right) \tag{35}$$

the even moments are $\langle x^{2k}\rangle = m^{2k}\exp(2k^2\sigma^2)$. The ratio $\frac{\langle x^{2k}\rangle}{\langle x^2\rangle^k} = \exp(2\sigma^2 k(k-1))$ varies thus with N as does $r_{2k,2}$ for $\sigma^2 \propto \ln(N)$. This suggests that the tails of the asymptotic

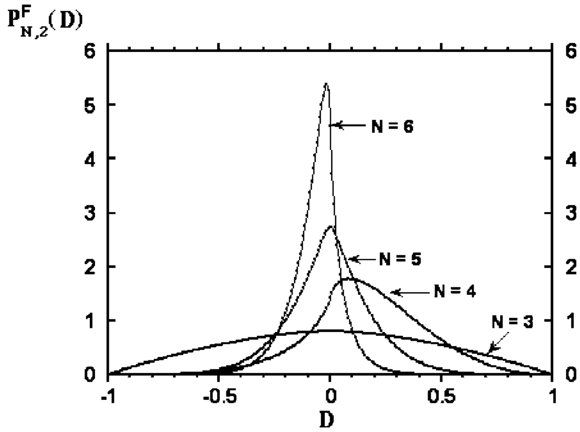

Figure 2. Determinant distribution $P^F_{N,2}(D)$ for $3 \leqslant N \leqslant 6$ from Monte Carlo simulations with 2×10^7 matrices.

determinant distribution behave as those of a lognormal distribution which is, however, not uniquely determined from its moments. As expected, the distribution of the linear statistics $V_0 = \log |D_F|$ is shown to be asymptotically Gaussian with a variance $\propto \ln(N)$ both for FTE(1) and FTE(2) (appendix B).

The method of the previous section is finally used in appendix C to calculate the distribution of the absolute value of the Vandermonde determinant: $\Delta(\lambda_1, \ldots, \lambda_N) = \prod_{1\leqslant j<k\leqslant N} (\lambda_j - \lambda_k)$ for the FTE(β)s.

4. Conclusion

The determinant distribution has been derived in a simple way for the fixed-trace ensembles FTE(β) of $N \times N$ random real-symmetric and Hermitian matrices from the known determinant probability densities of the GOE and of the GUE [21, 22] given either by a single Meijer's G-function or by a linear combination of two Meijer's G-functions according to the parity of N. All distributions dealt with in the present paper are expressed in terms of the following Meijer's G-functions: $G^{N,0}_{0,N}$ in the Gaussian case and $G^{N,0}_{N,N}$ in the fixed-trace ensemble case.

Appendix A. Some integrals over the surface of the *N*-dimensional unit sphere

As discussed in [25, 26], the integral

$$I^G_{N,\beta}(s) = \int_{\mathbb{R}^N} \prod_{k=1}^{N} \left\{ \mathrm{d}x_k |x_k|^s \exp\left(-\frac{x_k^2}{2}\right)\right\} \left| \prod_{1\leqslant i<j\leqslant N} (x_i - x_j)\right|^\beta \tag{A.1}$$

was used to derive the Mellin transform of the determinant distribution of the Gaussian RMEs ($\mathrm{Re}(s) > -1$) (see also [27]). The related integral over the surface of the N-dimensional unit sphere for FTE(β), namely

$$I^F_{N,\beta}(s) = \int_{\sum_{k=1}^N x_k^2 = 1} \prod_{k=1}^{N} \{\mathrm{d}x_k |x_k|^s\} \left| \prod_{1\leqslant i<j\leqslant N} (x_i - x_j)\right|^\beta \tag{A.2}$$

is related to integral (A.1) by

$$I^F_{N,\beta}(s) = \frac{I^G_{N,\beta}(s)}{2^{\frac{N_p+Ns}{2}-1}\Gamma\left(\frac{Ns+N_p}{2}\right)} \tag{A.3}$$

$\left(N_p = N + \beta\frac{N(N-1)}{2}\right)$ as shown by a change from Cartesian to spherical coordinates in equation (A.1). The difference between equations (A.3) and (17), which both relate Mellin transforms of Gaussian and of fixed-trace RMEs, comes from the fact that $\sigma = 1$ in equation (A.1) while $\sigma = 1/\sqrt{2}$ in equation (15). A similar calculation yields the following integral:

$$I^h_{N,\beta}(s) = \int \prod_{k=1}^{N} \{\mathrm{d}x_k |x_k|^s\} h\left(\sum_{k=1}^{N} x_k^2\right) |\Delta(x_1, \ldots, x_N)|^\beta \tag{A.4}$$

where $h(x)$ is proportional to a probability density and the integral runs over the domain of existence of $h(x)$. If the integral

$$H\left(\frac{Ns + N_p}{2}\right) = \int h(2z) z^{\frac{(Ns+N_p)}{2}-1}\,\mathrm{d}z \tag{A.5}$$

converges for a given s then integral (A.4) is

$$I^h_{N,\beta}(s) = I^G_{N,\beta}(s)\frac{H\left(\frac{Ns+N_p}{2}\right)}{\Gamma\left(\frac{Ns+N_p}{2}\right)}. \tag{A.6}$$

From the partition function of the Gaussian ensembles ([7], p 72), we get for instance

$$I^h_{N,\beta}(0) = \left(\frac{\sqrt{2\pi}}{\Gamma(1+\beta/2)}\right)^N \frac{H\left(\frac{N_p}{2}\right)}{\Gamma\left(\frac{N_p}{2}\right)} \prod_{j=1}^{N} [\Gamma(1+\beta j/2)] \tag{A.7}$$

whatever N while for $\beta = 1$ and $N = 2p + 1$, we obtain from [26]

$$I^h_{N,1}(s) = N!\left[\prod_{j=1}^{(N-3)/2} j!\right] 2^{(N+s)/2}\Gamma\left(\frac{1+s}{2}\right)\left[\frac{H\left(\frac{Ns+N_p}{2}\right)}{\Gamma\left(\frac{Ns+N_p}{2}\right)}\right]\prod_{j=1}^{(N-1)/2}\Gamma(j+s+1/2). \tag{A.8}$$

Appendix B. Linear statistics for FTE(β) with β = 1, 2

For Gaussian RMEs, the moment generating function $E^G_{N,\beta}(t)$ of the distribution $g^G_{N,\beta}(V_0)$ of the linear statistics $V_0 = \mathrm{Log}|D_G|$, the negative of the potential Q at the origin (section 1), is related to the Mellin transform of the even part of the determinant distribution (equation (6)) by [26]

$$E^G_{N,\beta}(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{V_0 t} g^G_{N,\beta}(V_0)\,\mathrm{d}V_0 = 2M^{G,+}_{N,\beta}(t+1). \tag{B.1}$$

A similar relation holds between $E^F_{N,\beta}(t)$ and $M^{F,+}_{N,\beta}(t+1)$ for FTEs, where now $V_0 = \mathrm{Log}|D_F|$. Using equation (17)

$$E^F_{N,\beta}(t) = E^G_{N,\beta}(t)\frac{\Gamma(N_p/2)}{\Gamma((Nt+N_p)/2)}. \tag{B.2}$$

Taking into account the definition of D_F (equation (13)) the means are related by

$$\langle V_0\rangle^F_{N,\beta} = \langle V_0\rangle^G_{N,\beta} - \frac{N}{2}\psi\left(\frac{N_p}{2}\right) + \frac{N}{2}\ln(N) \tag{B.3}$$

where $\langle V_0\rangle^G_{N,\beta}$, given by equation (42) of [26], is $\langle V_0\rangle^G_{\infty,\beta} = -N\left(\frac{1}{2}+\ln 2\right)$ for large N from which the asymptotic mean $\langle V_0\rangle^F_{\infty,\beta} = -\frac{N}{2}\ln N$ is obtained. The cumulants $\kappa^X_{N,\beta}(m) = \left[\frac{\mathrm{d}^m \ln(E^X_{N,\beta}(t))}{\mathrm{d}t^m}\right]_{t=0}$ $(X = F, G)$ $(m \geqslant 2)$ of the two considered RMEs are related through

$$\kappa^F_{N,\beta}(m) = \kappa^G_{N,\beta}(m) - \left(\frac{N}{2}\right)^m \psi^{(m-1)}\left(\frac{N_p}{2}\right) \tag{B.4}$$

where the polygamma functions are defined by $\psi^{(m-1)}(z) = \frac{\mathrm{d}^m \ln(\Gamma(z))}{\mathrm{d}z^m}$, $\psi(z) = \psi^{(0)}(z)$ [45]. The central moments $\mu^X_{N,\beta} = \left\langle (V_0 - \langle V_0\rangle)^m\right\rangle^X_{N,\beta}$ of orders 2 and 3 are equal to the cumulants of the same order $\kappa^X_{N,\beta}(m)$ $(m = 2, 3)$. The last term of equation (B.4) decreases as N^{m-2} for large N. The asymptotic variances are deduced from equation (45) of [26] and from equation (B.4):

$$\mu^F_{\infty,1} = \mu^G_{\infty,1} = \ln N \qquad \mu^F_{\infty,2} = \mu^G_{\infty,2} = \tfrac{1}{2}\ln N. \tag{B.5}$$

From equation (46) of [26], the asymptotic cumulants $\kappa^G_{N,\beta}(m)$ of order $m \geqslant 3$ are independent of N. From equation (B.4), the cumulants $\kappa^F_{N,\beta}(m)$ of order $m \geqslant 3$ are too independent of N for large N and the cumulants of order $m \geqslant 3$ of

$$U^F_{N,\beta} = \frac{V_0 + \frac{N\ln N}{2}}{\sqrt{\mu^F_{\infty,\beta}}} \tag{B.6}$$

are asymptotically equal to 0. The asymptotic distribution of $U^F_{N,\beta}$ is finally concluded to be a standard Gauss distribution as expected from general theorems on linear statistics [33–35].

Appendix C. Probability density of the absolute value of the Vandermonde determinant for FTE(β), β = 1, 2

The method of section 3 yields the distribution of the absolute value of the Vandermonde determinant

$$V = |\Delta(\lambda_1, \ldots, \lambda_N)| = \left|\prod_{1\leqslant j<k\leqslant N} (\lambda_j - \lambda_k)\right| \tag{C.1}$$

for the FTE(β)s. Stochastic representations have been obtained by Lu and Richards [44] for the distribution of the discriminant, the square of V, when $\lambda_1, \ldots, \lambda_N$ are iid Gaussian ($\beta = 0$), gamma or beta random variables. The Mellin transform of the distribution $P^G_{N,\beta}(V)$ of V is obtained easily from the Mehta integral (p 354 of [7]) to be

$$M^G_{N,\beta}(s) \propto \left(2^{-N(N-1)/2}\prod_{m=1}^{N} m^m\right)^{\frac{s}{2}} \prod_{j=1}^{N-1}\left(\prod_{k=1}^{j}\Gamma\left(\frac{s}{2}+\frac{\beta-1}{2}+\frac{k}{(j+1)}\right)\right) \tag{C.2}$$

(for $\mathrm{Re}(s+\beta-1) > -2/N$), that is

$$P^G_{N,\beta}(W) = \left(2\frac{(N!)^{1/2}}{(2\pi)^{N(N-1)/4}}\prod_{j=1}^{N}\left[\frac{j^{j\beta/2}\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(1+\frac{j\beta}{2}\right)}\right]\right) \times W^{\beta-1} G^{N(N-1)/2,0}_{0,N(N-1)/2}\left(W^2 \left| \frac{k}{(j+1)}, j = 1, \ldots, N-1, k = 1, \ldots, j\right.\right) \tag{C.3}$$

with $W = V\left(\frac{2^{N(N-1)/2}}{\prod_{m=1}^{N} m^m}\right)^{1/2}$. The moments of V obtained from the Mellin transform, equation (C.2)

$$\langle V^k\rangle^G_{N,\beta} = 2^{-kN(N-1)/4}\prod_{j=1}^{N}\left[\frac{\Gamma\left(\frac{\beta}{2}+1\right)\Gamma\left(\frac{j(k+\beta)}{2}+1\right)}{\Gamma\left(\frac{j\beta}{2}+1\right)\Gamma\left(\frac{k+\beta}{2}+1\right)}\right] \tag{C.4}$$

agree (with $\sigma = 1/\sqrt{2}$ in equation (4), for even k, with the moments of the discriminant D of the β-Hermite RME, an ensemble of tridiagonal random symmetric real matrices defined recently by Dumitriu and Edelman [48]. For $N = 3$:

$$P^G_{3,\beta}(W) = Z^{(2)}_{3,\beta} W^{\beta-1}\, G^{3,0}_{0,3}\left(W^2 \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}\right) \tag{C.5}$$

with

$$G^{3,0}_{0,3}\left(z \,\middle|\, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}\right) = \Gamma\left(\tfrac{1}{6}\right)\Gamma\left(\tfrac{1}{3}\right)\, {}_0F_2\left(\tfrac{5}{6}, \tfrac{7}{6}; z\right) z^{1/3} + \Gamma\left(-\tfrac{1}{6}\right)\Gamma\left(\tfrac{1}{6}\right)\, {}_0F_2\left(\tfrac{4}{3}, \tfrac{5}{3}; z\right) z^{1/2} + \Gamma\left(-\tfrac{1}{6}\right)\Gamma\left(-\tfrac{1}{3}\right)\, {}_0F_2\left(\tfrac{3}{2}, 2; z\right) z^{2/3}.$$

The relation between the probability distributions of V for the Gaussian ensembles and for the FTE(β)s is

$$P^G_{N,\beta}(V) = \frac{2}{\Gamma\left(\frac{N_p}{2}\right)}\int_{\left|\frac{V}{V_M}\right|^{\frac{2}{N(N-1)}}}^{\infty} P^F_{N,\beta}\left(\frac{V}{r^{N(N-1)/2}}\right) r^{N_p-1-\frac{N(N-1)}{2}}\exp(-r^2)\,\mathrm{d}r \tag{C.6}$$

where $V_M = [N(N-1)]^{-N(N-1)/4}\prod_{m=1}^{N} m^{m/2}$ is the maximum value of V over the surface of the unit N-dimensional sphere. From equation (C.6), we derive

$$M^F_{N,\beta}(s) = \frac{\Gamma\left(\frac{N_p}{2}\right)}{\Gamma\left(\frac{N_p}{2} + \frac{N(N-1)}{4}(s-1)\right)} M^G_{N,\beta}(s) \tag{C.7}$$

and thus

$$\langle V^k\rangle^F_{N,\beta} = \frac{\Gamma\left(\frac{N_p}{2}\right)}{\Gamma\left(\frac{N_p}{2} + \frac{N(N-1)k}{4}\right)}\langle V^k\rangle^G_{N,\beta} \tag{C.8}$$

with $\langle V^k\rangle^G_{N,\beta}$ given by equation (C.4). Defining

$$Y = V\left(\frac{[N(N-1)]^{N(N-1)/4}}{\prod_{m=1}^{N} m^{m/2}}\right) \tag{C.9}$$

we obtain finally the distribution of Y $(0 < Y < 1)$:

$$P^F_{N,\beta}(Y) = 2Z^{(4)}_{N,\beta} Y^{\beta-1} G^{\frac{N(N-1)}{2},0}_{\frac{N(N-1)}{2},\frac{N(N-1)}{2}}\left(Y^2 \,\middle|\, \begin{matrix}\frac{1}{N-1} + \frac{2(m-1)}{N(N-1)}, m = 1,\ldots,\frac{N(N-1)}{2}\\ \frac{k}{(j+1)}, j = 1,\ldots,N-1, k = 1,\ldots,j\end{matrix}\right) \tag{C.10}$$

where $Z^{(4)}_{N,\beta}$ is calculated from the condition $M^F_{N,\beta}(1) = 1$. As expected, the parameter β appears only in the exponent of the power multiplying Meijer's G-function in both distributions (C.3) and (C.10). The distributions of Y $(0 < Y < 1)$ for $N = 2, 3, 4$ are given by

$$P^F_{2,\beta}(Y) = \frac{2\Gamma(1+\beta/2)}{\Gamma\left(\frac{1+\beta}{2}\right)} Y^{\beta-1} G^{1,0}_{1,1}\left(Y^2 \,\middle|\, \begin{matrix}1\\ \frac{1}{2}\end{matrix}\right) = \frac{2\Gamma(1+\beta/2)}{\sqrt{\pi}\,\Gamma\left(\frac{1+\beta}{2}\right)} \times \frac{Y^\beta}{(1-Y^2)^{1/2}} \tag{C.11}$$

$$P^F_{3,\beta}(Y) = \frac{2\Gamma\left(\frac{5}{6}+\frac{\beta}{2}\right)\Gamma\left(\frac{7}{6}+\frac{\beta}{2}\right)}{\Gamma\left(\frac{1}{3}+\frac{\beta}{2}\right)\Gamma\left(\frac{2}{3}+\frac{\beta}{2}\right)} Y^{\beta-1} G^{2,0}_{2,2}\left(Y^2 \,\middle|\, \begin{matrix}\frac{5}{6}, \frac{7}{6}\\ \frac{1}{3}, \frac{2}{3}\end{matrix}\right) = c_\beta Y^{\beta+\frac{1}{3}} F\left(\tfrac{5}{6}, \tfrac{1}{2}; 1; 1-Y^2\right) \tag{C.12}$$

$$P_{4,\beta}^{F}(Y) = 2Z_{4,\beta}^{(4)}\, Y^{\beta-1} G_{3,3}^{3,0}\left(Y^2 \,\middle|\, \begin{matrix} \frac{5}{6}, 1, \frac{7}{6} \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \end{matrix}\right)$$
$$= d_\beta \left[\begin{matrix} Y^{\beta-\frac{1}{2}}\,{}_3F_2\left(\frac{1}{12}, \frac{1}{4}, \frac{5}{12}; \frac{1}{2}, \frac{3}{4}; Y^2\right) - 6Y^\beta\,{}_3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; \frac{3}{4}, \frac{5}{4}; Y^2\right) \\ + Y^{\beta+\frac{1}{2}}\,{}_3F_2\left(\frac{7}{12}, \frac{3}{4}, \frac{11}{12}; \frac{5}{4}, \frac{3}{2}; Y^2\right) \end{matrix}\right]. \tag{C.13}$$

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